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The Free Energy and Entropy of Schwarzschild Black Hole Due to Scalar Field

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Using the thin film brick-wall model,taking into account the effect of the generalized uncertainty principle on the equation of the density of the states, we calculate the free energy and entropy of schwarzschild black hole due to scalar field, we obtain the entropy proportional to the event horizon area without cutoff. This implies that quantum theory of gravity can remove the divergence of the state density on the event horizon and avoid the cutoff in the original brick-wall model, these results also mean that the thin film brick-wall model is universal.

KEY WORDS: entropy; generalized uncertainty principle; thin film brick-wall model.

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1. INTRODUCTION

The discovery of the Hawking radiation confirms the conjecture that the blackhole entropy is proportional to the event horizon area (Bekenstein, 1973; Gibbons and Hawking, 1977; Hawking, 1975). But it remains a problem that how to explain the statistical origin of the entropy. Meanwhile many ways of calculating the entropy has emerged (Cai *et al.*, 1998; Cognola and Lecca, 1998; Hochberg *et al.*, 1993; Li and Zhao, 2000; Padmanaban, 1989). The most frequently used method among them is the brick-wall method advanced by t'Hooft (1985), in which the entropy of a black hole is identified with the entropy of the thermal bath of the quantum field outside the event horizon. The method is used to study the statistical property of Scalar field and Dirac field in various black hole (Jing and Yan, 2000; Solodukhin, 1995; Zhang and Zhao, 2002; Zhao *et al.*, 2002). However it is found that the quantum state density is divergent near event horizons. Therefore in the brick-wall model the ultraviolet cutoff is introduced by hand and it looks unnatural.

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The recent works show that quantum theory of gravity can transform the quantum uncertainty principle into generalized uncertainty principle (Chang *et al.*, 2002; Kempt *et al.*, 1995; Li, 2002a,b). Introducing the generalized uncertainty principle to calculate black hole entropy, the state density and entropy near event horizon are convergent, and the cutoff in the brick-wall model can be removed (Liu *et al.*, 2004, 2003; Zhang and Zhao, 2004; Zhang *et al.*, 2004). In this paper, we take into account the effect of the generalized uncertainty principle on the equation of the density of the states, and calculate the free energy and entropy of schwarzschild black hole due to scalar field by using the thin film brick-wall model, we obtain the conclusion that the entropy is proportional to the event horizon area.

2. THE METRIC OF SCHWARZSCHILD

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

The position, temperature and area of the event horizon are given respectively by

$$r_h = 2M, T_0 = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}, A_h = 4\pi r_h^2 \tag{2}$$

where κ is the surface gravity at the horizon.

Substituting Eq. (1) into the Klein-Gordon equation of scalar field as follows

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = m_0^2\Phi \tag{3}$$

where the det $|g_{\mu\nu}| = -r^4\sin^2\theta$

$$g^{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} \tag{4}$$

$$\frac{1}{r^2\sin\theta}\partial_\mu(r^2\sin\theta g^{\mu\nu}\partial_\nu\Phi) - m_0^2\Phi = 0 \tag{5}$$

According to the symmetric property of the space-time, the wave function of the scalar field can be separately expressed as

$$\Phi = e^{-i\omega t}R(r)Y(\theta,\varphi) \tag{6}$$

where R(r) is only a function of r, ω is the energy of the excitated state of the scalar field. We can get the radial equation of the wave function and the equation of spherical function as follows.

$$\left(1-\frac{2M}{r}\right)^{-1}\omega^2 R(r)+\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(1-\frac{2M}{r}\right)\frac{d}{dr}R(r)\right] - \left(m_0^2+\frac{l(l+1)}{r^2}\right)R(r)=0 \tag{7}$$

$$\frac{\partial^2 Y}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+l(l+1)Y=0 \tag{8}$$

where $l(l+1)$ is the separation constant.

We apply Wentzel-Kramers-Brillouin (WKB) approximation in Eq. (7), $R(r)$ should be written as $R(r)\sim e^{iS(r)}$ and then by substituting it into Eq. (7), the radial wave number can be obtained as

$$p_r^2=k_r^2=\left(\frac{\partial S(r)}{\partial r}\right)^2=\left[\left(1-\frac{2M}{r}\right)^{-1}\omega^2-\left(m_0^2+\frac{l(l+1)}{r^2}\right)\right]\left(1-\frac{2M}{r}\right)^{-1} \tag{9}$$

We set

$$Y(\theta,\varphi)=e^{iz(\theta,\varphi)} \tag{10}$$

We obtain

$$\left(\frac{\partial Z}{\partial\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial Z}{\partial\varphi}\right)^2-l(l+1)=0 \tag{11}$$

$$p_\theta=\frac{\partial Z}{\partial\theta},\quad p_\varphi=\frac{\partial Z}{\partial\varphi} \tag{12}$$

thus we obtain the square modulus of momentum

$$P^2=p_i p^i=g^{rr}p_r^2+g^{\theta\theta}p_\theta^2+g^{\varphi\varphi}p_\varphi^2=\frac{\omega^2}{1-\frac{2M}{r}}-m_0^2 \tag{13}$$

Recently, the generalized uncertainty relation(GUP) followed by the generalized uncertainty principle is given by (Chang *et al.*, 2002; Kempt *et al.*, 1995; Li, 2002a,b)

$$\Delta x\Delta p\geq\hbar+\frac{\lambda}{\hbar}(\Delta p)^2 \tag{14}$$

where λ is a constant displaying the gravitationally induced correction to the quantum uncertainty relation(QUP), and it is of the order of the Planck area. Equation (14) implies that the uncertainty of position should not be infinitesimal, and its minimal value is $2\sqrt{\lambda}$ (Liu *et al.*, 2004; Zhang *et al.*, 2004).

As is well known, the number of quantum states in the integrals volume $dV\, d^3P$ is $\frac{dV\, d^3P}{(2\pi\hbar)^3}$, which can be understood as follows: since the uncertainty relation $\Delta x \Delta p \sim 2\pi\hbar$, one quantum state corresponds to a 'cell' of volume $(2\pi\hbar)^3$ in the phase space. According to the quantum theory of gravity, using GUP, the number of quantum states in the integrals $dV\, d^3P$ is given by

$$dN = \frac{d^3\overrightarrow{x}\, d^3\overrightarrow{p}}{(2\pi\hbar)^3(1+\lambda p^2)^3} \tag{15}$$

The generalized commutation relation is

$$[\widehat{x}, \widehat{p}] = i\hbar(1+\lambda p^2) \tag{16}$$

among the generalized uncertainty relation (14), the expression of the density of states (15) and the generalized commutation relation (16), the Planck constant $\hbar$ in the standard Heisenberg principle, has been replaced by a new parameter $\hbar' = \hbar(1+\lambda p^2)$. We take the simplest functional form of the Planck unit system $(C = G = K_B = \hbar = 1)$.

Substituting Eq. (13) into Eq. (15), we obtain the number of quantum states in the integrals $d^3\overrightarrow{x}\, d^3\overrightarrow{p}$ outside the event horizon of schwarzchild black hole

$$dN = \frac{d^3\overrightarrow{x}\, d^3\overrightarrow{p}}{(2\pi\hbar)^3\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}} - m_0^2\right)\right]^3} \tag{17}$$

Near the horizon, $(1-\frac{2M}{r}) \to 0$, $\frac{\omega^2}{1-\frac{2M}{r}} \to \infty$, then we can understand that the divergence of the quantum state density based on the QUP can be removed by using the GUP, thus the free energy and entropy of the quantum field near the horizon can be calculated without any cutoff.

3. THE FREE ENERGY AND THE ENTROPY OF SCHWARZCHILD BLACK HOLE

With locating the brick-wall at ε from the event horizon of the schwarzchild black hole, at first, we use the Eq. (17) to count the quantum states between the horizon and the brick-wall. Setting ε is an infinitesimal parameter and corresponds to the minimal length $2\sqrt{\lambda}$, then we have

$$\int_{r_h}^{r_h+\varepsilon} \sqrt{g_{rr}}dr = \int_{r_h}^{r_h+\varepsilon} \frac{dr}{\sqrt{g_{tt}}} \approx \int_{r_h}^{r_h+\varepsilon} \frac{dr}{\sqrt{2k(r-r_h)}} = \sqrt{\frac{2\varepsilon}{\kappa}} = 2\sqrt{\lambda} \tag{18}$$

According to Eq. (17), the number of quantum states energy less than ω is given by

$$
\begin{aligned}
\Gamma(\omega) &= \frac{1}{(2\pi)^3}\int \frac{dr d\theta d\varphi dp_r\, dp_\theta\, dp_\varphi}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}-m_0^2\right)\right]^3} = \frac{1}{(2\pi)^3}\int \frac{dr\, d\theta\, d\varphi}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}-m_0^2\right)\right]^3} \\
&\quad\times \int \frac{2}{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}}\left[\left(1-\frac{2M^{-1}}{r}\right)\omega^2-\left(m_0^2+\frac{l(l+1)}{r^2}\right)\right]^{\frac{1}{2}} dp_\theta\, dp_\varphi \\
&= \frac{1}{(2\pi)^3}\int \frac{dr\, d\theta\, d\varphi}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}-m_0^2\right)\right]^3} \\
&\quad\times \int \frac{2}{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}}\left[\left(1-\frac{2M}{r}\right)^{-1}\omega^2-\left(m_0^2+\frac{p_\theta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\theta}\right)\right]^{\frac{1}{2}} dp_\theta\, dp_\varphi \\
&= \frac{2}{3\pi}\int_{r_h}^{r_h+\varepsilon} \frac{r^2\left[\left(1-\frac{2M}{r}\right)^{-1}\omega^2-m_0^2\right]^{\frac{3}{2}}}{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}-m_0^2\right)\right]^3}\, dr \\
&= \frac{2}{3\pi}\int_{r_h}^{r_h+\varepsilon} \frac{r^2\left[\omega^2-m_0^2\left(1-\frac{2M}{r}\right)\right]^{\frac{3}{2}}}{\left(1-\frac{2M}{r}\right)^2\left[1+\lambda\left(\frac{\omega^2-m_0^2\left(1-\frac{2M}{r}\right)}{1-\frac{2M}{r}}\right)\right]^3}\, dr
\end{aligned} \tag{19}
$$

where the integration goes over those of p_θ and p_φ for which the argument of the square root is positive, and because ε is an infinitesimal parament, near the horizon, $(1-\frac{2M}{r}) \to 0$, then the above integration (19) can be simplied as follows

$$
\Gamma(\omega) = \frac{2\omega^3}{3\pi}\int_{r_h}^{r_h+\varepsilon} \frac{r^2}{\left(1-\frac{2M}{r}\right)^2\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}\right)\right]^3} dr \tag{20}
$$

Surprisingly, we do not take small mass approximation intentionally, the result is due to our restricting of integration region to an thin film near the horizon.

According to the theory of canonical ensemble, the free energy of system can be given by

$$
F = \frac{1}{\beta}\sum_{\omega} \ln(1-e^{-\beta\omega}) \tag{21}
$$

In terms of the semi-classical theory and assuming that the energy is continuous, we replace the sum by integration, and substitute Eq. (20) into Eq. (21), then

$$\begin{aligned}\beta F &= \int_0^{\infty} d\Gamma(\omega)\ln(1-e^{-\beta\omega}) \qquad (22)\\ &= \Gamma(\omega)\ln(1-e^{-\beta\omega})|_0^{\infty} - \int_0^{\infty}\frac{\Gamma(\omega)e^{-\beta\omega}}{1-e^{-\beta\omega}}\beta d\omega\\ &= -\beta\int_0^{\infty}\frac{\Gamma(\omega)}{e^{\beta\omega}-1}d\omega\\ &= -\frac{2\beta}{3\pi}\int_{r_h}^{r_h+\varepsilon}\frac{r^2}{\left(1-\frac{2M}{r}\right)^2}dr\int_0^{\infty}\frac{\omega^2}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}\right)\right]^3(e^{\beta\omega}-1)}d\omega\end{aligned}$$

The entropy of quantum field in the thin film from the horizon r_h to $r_h+\varepsilon$ is given by

$$\begin{aligned}S &= \beta^2\frac{\partial F}{\partial\beta} \qquad (23)\\ &= \frac{2\beta^2}{3\pi}\int_{r_h}^{r_h+\varepsilon}\frac{r^2}{\left(1-\frac{2M}{r}\right)^2}dr\int_0^{\infty}\frac{\omega^4 e^{\beta\omega}}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}\right)\right]^3(e^{\beta\omega}-1)^2}d\omega\\ &= \frac{2\beta}{3\pi}\int_{r_h}^{r_h+\varepsilon}\frac{r^2}{\left(1-\frac{2M}{r}\right)^2}dr\int_0^{\infty}\frac{\omega^4 d(\beta\omega)}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}\right)\right]^3(e^{\beta\omega}-1)(1-e^{-\beta\omega})}\end{aligned}$$

By using the following inequalities

$$e^{\beta\omega}-1>\beta\omega,\ 1-e^{-\beta\omega}>\frac{\beta\omega}{1+\beta\omega} \qquad (24)$$

we have

$$\begin{aligned}S &< \frac{2\beta^{-3}}{3\pi}\int_{r_h}^{r_h+\varepsilon}\frac{r^2}{\left(1-\frac{2M}{r}\right)^2}dr\int_0^{\infty}\frac{\beta^2\omega^2+\beta^3\omega^3}{\left[1+\lambda\left(\frac{\omega^2}{1-\frac{2M}{r}}\right)\right]^3}d(\beta\omega)\\ &= \frac{\lambda^{-\frac{3}{2}}}{24}\int_{r_h}^{r_h+\varepsilon}\frac{r^2}{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}}dr+\frac{\beta}{6\pi\lambda^2}\int_{r_h}^{r_h+\varepsilon}r^2dr \qquad (25)\end{aligned}$$

Instituting Eqs. (2) and (18) into above inequality, we obtain the upper bound of the entropy of the schwarzschild black hole

$$S \approx \frac{\lambda^{-\frac{3}{2}}}{24}r_h^2\times 2\sqrt{\lambda}+\frac{\beta}{6\pi\lambda^2}r_h^2\varepsilon = \frac{9A_h}{48\lambda} \qquad (26)$$

4. CONCLUSION

By using the WKB approximation, we solve the wave equation and obtain the entropy from the thin film near horizon proportional to the horizon area. In the above analysis, we find that gravity can change the density of the quantum states, especially this change is essential near the horizon. then the divergence appearing in the brick wall model is removed, without any cutoff. Our conclusion also means that the entropy of the black hole is related to the existence of the horizon, and it is the intrinsic property of the horizon.

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